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The construction of finer compact topologies (extended abstract of talk presented at the Dagstuhl Seminar 04351)[1]

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Definition 1 (compare [2,5,8]) A topological space is called a *KC-space* provided that each compact set is closed. A topological space is called a *US-space* provided that each convergent sequence has a unique limit.

Remark 1 Each Hausdorff space (= T_2-space) is a KC-space, each KC-space is a US-space and each US-space is a T_1-space (that is, singletons are closed); and no converse implication holds, but each first-countable US-space is a Hausdorff space.

Definition 2 A compact topology on a set X is called *maximal compact* provided that it is not strictly contained in a compact topology on X.

Remark 2 [4] A topological space is maximal compact if and only if it is a KC-space that is also compact. (These spaces will be called compact KC-spaces in the following.)

Example 1 A standard example of a maximal compact topology that is not a Hausdorff topology is given by the one-point-compactification of the set of rationals equipped with its usual topology.

Indeed we next note that maximal compact spaces can be anti-Hausdorff (= irreducible).

A nonempty subspace S of a topological space is called *irreducible* if each pair of nonempty open sets of S intersects. Furthermore a topological space

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X called a *Fréchet space* provided that for every $A \subseteq X$ and every $x \in \overline{A}$ there exists a sequence of points of A converging to x.

Lemma 1 *Each Fréchet US-space X is a KC-space.*

Example 2 (van Douwen) [7] There exists a countably infinite compact Fréchet US-space that is anti-Hausdorff. By the preceding lemma that space is a KC-space and hence maximal compact. Thus there exists an infinite maximal compact space that is irreducible.

On the other hand, by the result cited above each first-countable maximal compact (T_1-)topology satisfies the Hausdorff separation condition.

Let us recall that a topological space is called *strongly sober* provided that the set of limits of each ultrafilter is equal to the closure of some unique singleton. Of course, each compact Hausdorff space satisfies this condition.

A topological space is said to be *locally compact* provided that each of its points has a neighborhood base consisting of compact sets. (Note that a locally compact KC-space is a regular Hausdorff space.) Furthermore a subset of a topological space is called *saturated* provided that it is equal to the intersection of its open supersets.

Example 3 Each locally compact strongly sober topological space (X, τ) possesses a finer compact Hausdorff topology; just take the supremum of τ and its dual topology. By definition, the latter topology is generated by the subbase $\{X \setminus K : K \text{ is compact and saturated in } X\}$ on X.

No characterization seems to be known of those topologies that possess a finer compact Hausdorff topology.

Main Problem 1 While it is known that each compact topology is contained in a compact T_1-topology (just take the supremum of the given topology with the cofinite topology), the question whether each compact topology is contained in a compact KC-topology (that is, is contained in a maximal compact topology) seems still to be open. Apparently that question was first asked by Cameron, but remained unanswered (see [1]).

Example 4 Each infinite topological space X with a point x possessing only cofinite neighborhoods is contained in a maximal compact topology: Consider the one-point-compactification X_x of $X \setminus \{x\}$ where $X \setminus \{x\}$ is equipped with the discrete topology and x acts as the point at infinity.

Generalization of Main Problem 2 Is each compact topology the continuous image of a maximal compact topology?

Remark 3 [6] It is known that a compact space need not be the continuous image of a compact T_2-space. Indeed a KC-space Y that is the continuous image of a compact T_2-space X is a T_2-space.

We have the following positive partial answer to Cameron's question:

Theorem 1 *Let (X, τ) be a compact space. Then there is a compact topology τ' finer than τ such that (X, τ') is a US-space.*

Remark 4 It is possible to strengthen the latter result further to the statement that each compact topology is contained in a compact topology with respect to which each compact countable set is closed.

We recall that a topological space is called *sequentially compact* provided that each of its sequences has a convergent subsequence.

A modification of our arguments allows us to answer positively the variant of the main problem (also due to Cameron) formulated for sequential compactness instead of compactness:

Theorem 2 *Each sequentially compact topology τ on a set X is contained in a sequentially compact topology τ' that is maximal among the sequential compact topologies on X.*

Recall that a topological space is called *sober* provided that every irreducible closed set is the closure of some unique singleton. Clearly each Hausdorff space is sober.

The following statement is known under the name of Hofmann-Mislove Theorem (compare [3]):

Let $\{K_i : i \in I\}$ be a filterbase of (nonempty) compact saturated subsets of a sober space X. Then $\bigcap_{i \in I} K_i$ is nonempty, compact, and saturated, too; and an open set U contains $\bigcap_{i \in I} K_i$ iff U contains K_i for some $i \in I$.

Lemma 2 *Let (X, τ) be a compact topological space such that each filter-base consisting of compact subsets has a nonempty intersection. Then τ is contained in a maximal compact topology τ'.*

With the help of the Hofmann-Mislove Theorem we are able to prove the following result:

Proposition 1 *Let (X, τ) be a compact sober T_1-space. Then τ is contained in some maximal compact topology τ'.*

Remark 5 Let us observe that the maximal compact topology τ' obtained above will be sober, since the only irreducible sets with respect to the coarser topology τ are the singletons. Van Douwen's example mentioned earlier shows that a maximal compact topology need not be (contained in) a compact sober topology.

The following variant of Cameron's problem seems also to be of interest.

Problem 3 Which (compact) T_1-topologies are the infimum of a family of maximal compact topologies?

Evidently the cofinite topology on an infinite set X is the infimum of the family of maximal compact Hausdorff topologies of the one-point-compactifications X_x (where $x \in X$) that we have defined above.

For compact sober T_1-topologies we have the following partial result.

Recall that a topological space X is called *sequential* provided that a set $A \subseteq X$ is closed if and only if together with any sequence it contains all its limits in X.

Proposition 2 *Each compact sober T_1-space which is locally compact or sequential is the infimum of a family of maximal compact topologies.*

Example 5 Let Y be an uncountable set and let $-\infty$ and ∞ be two distinct points not in Y. Set $X = Y \cup \{-\infty, \infty\}$. Each point of Y is supposed to be isolated. The neighborhoods of ∞ are the cofinite sets containing ∞ and the neighborhoods of $-\infty$ are the cocountable sets containing $-\infty$. Clearly (X, τ) is a compact sober T_1-space. Observe that the topology τ' generated by the subbase $\{\{-\infty\}\} \cup \tau$ clearly yields a compact T_2-topology finer than

τ. One shows that τ' is the only maximal compact topology (strictly) finer than τ. So the conclusion of Proposition 2 does not hold for the space (X, τ).

References

[1] Cameron, D.E.: A survey of maximal topological spaces, Topology Proc. **2** (1977), 11–60.

[2] Fleissner, W.G.: A T_B-space which is not Katetov T_B, Rocky Mountain J. Math. **10** (1980), 661–663.

[3] Keimel, K. and Paseka, J.: A direct proof of the Hofmann-Mislove theorem, Proc. Amer. Math. Soc. **120** (1994), 301–303.

[4] Ramanathan, A.: Minimal-bicompact spaces, J. Ind. Math. Soc. (N.S.) **12** (1948), 40–46.

[5] Reed, M.J.: Hausdorff-like separation properties and generalizations of the first countability axiom, Tamkang J. Math. **5** (1974), 197–201.

[6] Stone, A.H.: Compact and compact Hausdorff, in: Aspects of Topology, pp. 315–324, London Math. Soc., Lecture Note Ser. **93**, Cambridge Univ. Press, Cambridge, 1985.

[7] van Douwen, E.K.: An anti-Hausdorff Fréchet space in which convergent sequences have unique limits, Topology Appl. **51** (1993), 147–158.

[8] Wilansky, A.: Between T_1 and T_2, Amer. Math. Monthly **74** (1967), 261–266.